Transition Maths and Algebra with Geometry

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Lecture Notes
Electrical and Computer Engineering

PROGRAM ROZWOJOWY POLITECHNIKI WARSZAWSKIEJ

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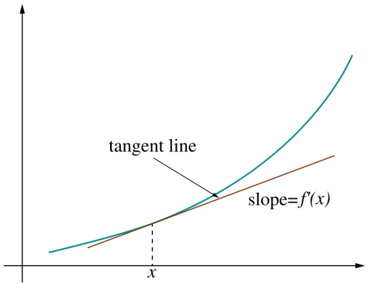
KAPITAŁ LUDZKI
CZŁOWIEK – NAJLEPSZA INWESTYCJA!

Contents

Tangent lines

Consider a line tangent to the graph of a function f at some point x_0 belonging to its domain.



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Tangent lines

If the tangent line $y = a \cdot x + b$ exists (because in general it doesn't have to) then the slope a of the tangent line measures how a function changes around x_0.

Question

Given a function $y = f(x)$ and a point x_0 from its domain how can we compute the slope a of the tangent to $y = f(x)$ at x_0?

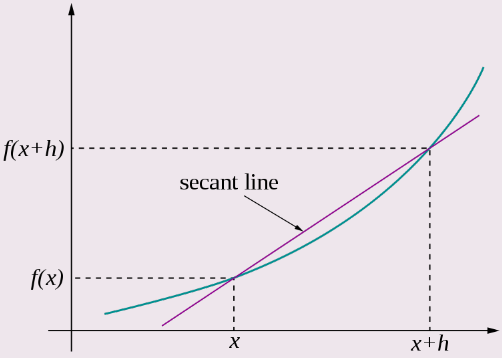
Tangent lines

Instead of a tangent line, consider a secant line passing through $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$. Slope of this line is given by

$$\frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Tangent lines

Secant line



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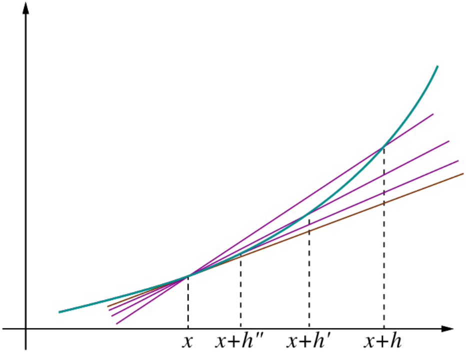
Tangent lines

Observe that as h goes to 0 the secant line approaches to the tangent line at x_0.



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Tangent lines

Question

Given a function $y = f(x)$ and a point x_0 from its domain how can we compute the slope a of the tangent to $y = f(x)$ at x_0?

Answer: Consider the slopes of secants

$$\frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

and calulate their limit as h tends to 0.

Derivative

Definition

The derivative of a function $y = f(x)$ is a function $y = f'(x)$ whose value at x_0 is defined by

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Warning

Note that f' may not exist at some points belonging to the domain of f.

Definition

If f' exists at a point x_0 then we say that the function $y = f(x)$ is *differentiable* at x_0.

Derivative: examples

Consider a function $f(x) = a \cdot x + b$ and find its derivative.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} =$$

$$\lim_{h \to 0} \frac{a \cdot (x_0 + h) + b - (a \cdot x_0 + b)}{h} = \lim_{h \to 0} \frac{a \cdot h}{h} = a.$$

Derivative: examples

Consider a function $f(x) = x^2$ and find its derivative.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} =$$

$$\lim_{h\to 0} \frac{x_0^2 + 2x_0 \cdot h + h^2 - x_0^2}{h} = \lim_{h\to 0} \frac{2x_0 \cdot h + h^2}{h} = \lim_{h\to 0}(2 \cdot x_0 + h) = 2x_0.$$

Derivative: examples

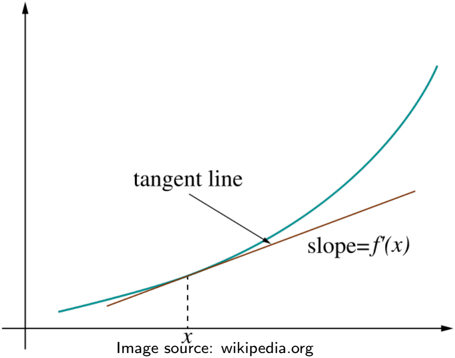
Consider a function $f(x) = |x|$. Its derivative at $x_0 = 0$ doesn't exist because the limit

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

doesn't exist.

Geometrical interpretation

The derivative $f'(x_0)$ at x_0 is equal to the slope of the tangent line to the curve $y = f(x)$ at x_0.



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Geometrical interpretation

If the tangent line to the curve $y = f(x)$ at point x_0 is given by an equation $y = a \cdot x + b$ then

$$a = f'(x_0),$$
$$b = -f'(x_0)x_0 + f(x_0).$$

Equation of tangent

Equation of the tangent line to the curve $y = f(x)$ at point x_0 is given by

$$y = f'(x_0) \cdot (x - x_0) + f(x_0)$$

One-sided derivatives

Definition

Let $y = f(x)$ be a function. A left-hand (right-hand) derivative f'_- (resp. f'_+) at x_0 is defined by

$$f'_-(x_0) = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h},$$

$$f'_+(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Remark

There are functions whose left-hand and right-hand derivative exist but whose proper derivative doesn't (e.g. $y = |x|$).

One-sided derivatives

Theorem

Let $y = f(x)$ be a function whose left-hand and right-hand derivatives exist at x_0 and

$$f'_-(x_0) = f'_+(x_0).$$

Then $f'(x_0)$ exists and

$$f'(x_0) = f'_-(x_0) = f'_+(x_0).$$

Differentiation and continuity

Theorem

If a function $y = f(x)$ is differentiable at x_0 then it is continuous at x_0.

Remark

If a function is continuous it doesn't mean it is differentiable (e.g. $y = |x|$).

Contents

Differentiation rules

Theorem

- $(c)' = 0$ (derivative of a constant function is zero),
- $[c \cdot f(x)]' = c \cdot f'(x)$,
- $[f(x) + g(x)]' = f'(x) + g'(x)$,
- $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ (product rule),
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$ (quotient rule),
- $[f(g(x))]' = f'(g(x)) \cdot g'(x)$ (chain rule).

Differentiating inverse functions

Theorem

Let $y = f(x)$ be a 1-1 function and let f be differentiable at x_0 with $f'(x_0) \neq 0$. Then the inverse function $x = f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Differentiating inverse functions

Example: Consider $y = x^2$ defined for $D = \{x \in \mathbb{R} \mid x \geq 0\}$. It is $1-1$ and $f'(x) = 2 \cdot x$. For any $x_0 > 0$ we have $f'(x_0) \neq 0$. So for any $x_0 > 0$ the inverse $f^{-1}(y) = \sqrt{y}$ is differentiable at $y_0 = f(x_0) = x_0^2$ and

$$(f^{-1})'(y_0) = (\sqrt{y})'_{y=y_0} = \frac{1}{f'(x_0)} = \frac{1}{2x_0} = \frac{1}{2\sqrt{y_0}}$$

Derivatives of elementary functions

Theorem

- $(x^n)' = n \cdot x^{n-1}$ for $n \in \mathbb{Z} \setminus \{0\}$,
- $(x^\alpha)' = \alpha \cdot x^{\alpha-1}$ for $\alpha \in \mathbb{R}$ and $x > 0$,
- $(\sin x)' = \cos x$,
- $(\cos x)' = -\sin x$,
- $(\tan x)' = \frac{1}{\cos^2 x}$ for $x \neq \frac{\pi}{2} + k\pi$,
- $(\cot x)' = -\frac{1}{\sin^2 x}$ for $x \neq k\pi$,
- $(e^x)' = e^x$,
- $(a^x)' = a^x \ln a$ for $a > 0$ and $a \neq 1$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $|x| < 1$,
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $|x| < 1$,

Derivatives of elementary functions

Theorem

- $(\arctan x)' = \frac{1}{1+x^2}$,
- $(\ln x)' = \frac{1}{x}$ for $x > 0$,
- $(\log_a x)' = \frac{1}{x \ln a}$ for $x > 0$ and $0 < a \neq 1$.

Examples

Calculate

$$\arctan(\ln x)',$$
$$(e^x \cdot \sin x)',$$
$$\left(\frac{x^2+1}{x^2-1}\right)'.$$

Contents

Derivatives of higher order

Definition

Second derivative of $y = f(x)$ is a function f'' defined by

$$f''(x) = (f'(x))'.$$

Generally, we define n-th derivative of f by

$$f^{(n)} = [f^{(n-1)}]'$$

Notation

$$f', f'', f''', \dots, f^{(n)}$$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}$$

Derivatives of higher order

Warning

If a function is n times differentiable it does not mean it is $n+1$ times differentiable. Consider a function $f(x) = x|x|$.

Linear approximation

Definition of derivative (once again!)

A derivative $f'(a)$ at point a of the function $y = f(x)$ is is a number defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We can restate the definition as follows.

Equivalent definition of derivative

A derivative $f'(a)$ at point a of the function $y = f(x)$ is a number satisfying

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0.$$

Linear approximation

Linear approximation

If parameter h is very close to 0 then

$$f(a+h) \approx f(a) + f'(a)h.$$

In other words, if a variable x is close to a then

$$f(x) \approx f(a) + f'(a)(x-a).$$

Example

Let $f(x) = sin(x)$ and $a = \pi$. If x is close to π then

$$\sin(x) \approx \sin(\pi) + \sin'(\pi) \cdot (x - \pi) = -x + \pi.$$

Higher order derivatives and approximation

Question

Can we use derivatives of higher order to approximate functions better?

Yes, we can!

Quadratic approximation

Quadratic approximation

Let $y = f(x)$ be a function. We say that $y = f(x)$ is *well approximated by a quadratic function* $ax^2 + bx + c$ around x_0 if

$$\lim_{h\to 0} \frac{f(x_0+h) - ah^2 - bh - c}{h^2} = 0.$$

Theorem

Let $y = f(x)$ be twice differentiable at x_0 and let the function $y = f(x)$ be well approximated by $a(x-x_0)^2 + b(x-x_0) + c$ around x_0. Then

$$a = \frac{f''(x_0)}{2} \qquad b = f'(x_0) \qquad c = f(x_0).$$

Quadratic approximation

Quadratic approximation

If parameter h is very close to 0 then

$$f(a+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2}h^2.$$

In other words, if a variable x is close to a then

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Quadratic approximation

Example

Example Let $f(x) = \cos x$ and $a = 0$. Then around $a = 0$ we have

$$\cos x \approx \cos(0) - \sin(0) \cdot x + \frac{-\cos(0)}{2} \cdot x^2 = 1 - \frac{x^2}{2}.$$

Polynomial approximation

Polynomial approximation

Let $y = f(x)$ be a function. We say that $y = f(x)$ is *well approximated by a polynomial* $p(x) = a_n x^n + \ldots + a_1 x + a_0$ *of degree* n around a point x_0 if

$$\lim_{h \to 0} \frac{f(x_0 + h) - p(h)}{h^n} = 0.$$

Theorem

Let $y = f(x)$ be n times differentiable at x_0 and let the function $y = f(x)$ be well approximated by a polynomial $p(x) = a_n(x - x_0)^n + \ldots + a_0$ around x_0. Then

$$a_k = \frac{f^{(k)}(x_0)}{k!} \text{ for } k = 0, \ldots, n.$$

Polynomial approximation

Example

Example Let $f(x) = \cos x$ and $a = 0$. Then around $a = 0$ we have

$$\cos x \approx 1,$$

$$\cos x \approx 1 - \frac{x^2}{2},$$

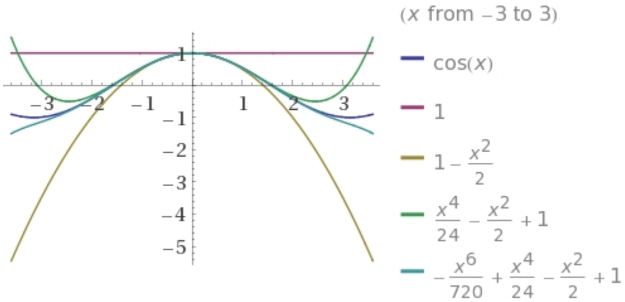
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!},$$

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!},$$

$$\ldots$$

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{2n!}.$$

Polynomial approximation



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Different notations

Let $y = f(x)$. Sometimes different symbols for derivatives are used:

$$\dot{y} \qquad \frac{dy}{dx} \qquad f' \qquad D_x f$$

Physics 101

Velocity and acceleration

Derivatives measure the rate of change of one variable with respect to the other. If $x = s(t)$ is a function describing a position of a point x at a given time t then

$$v(t) = s'(t) \text{ - velocity at a given time } t,$$
$$a(t) = v'(t) = s''(t) \text{ - acceleration at a given time } t.$$

Physics 101

Note

We can also consider the values of the function $s(t)$ to be more than 1-dimensional vectors, i.e.

$$\vec{x} = s(t).$$

If the function depends on one variable (in our case time t) then the differentiation is done coordinate wise.

Physics 101

Example

Let a position of a particle be described using the following function

$$s(t) = (a \cdot e^{b \cdot t} \cos(t), a \cdot e^{b \cdot t} \sin(t)) \text{ where } a, b \text{ are parameters.}$$

Find its velocity and acceleration vector at time $t = 1$.

Error approximation

Let's recall

For a function $y = f(x)$ if parameter h is very close to 0 then

$$f(a+h) \approx f(a) + f'(a)h.$$

In other words,

$$f(a+h) - f(a) \approx f'(a)h \qquad \Delta f \approx f'(a)\Delta x.$$

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Newton's method

Suppose we are given a function $f(x)$ and we want to find a point (a good estimate will suffice) x such that $f(x) = 0$.

An application?

Calculate a good approximation of e.g. $\sqrt{3}$.

Newton's method

Suppose we are given a function $f(x)$ and we want to find a point (a good estimate will suffice) x such that $f(x) = 0$.

An application?

Calculate a good approximation of e.g. $\sqrt{3}$.

Answer: consider $f(x) = x^2 - 3$.

1. Guess a rough estimate of a zero. Say $x = 2$,
2. consider the tangent line to the curve $f(x)$ at $x = 2$. Since $f'(x) = 2x$ the formula for the tangent line is given $y = 4x - 7$,
3. the point at which the tangent line crosses the x-axis should be a better estimate of zero. Indeed, the solution is $x = \frac{7}{4}$.
4. repeat the whole procedure (2)-(4) until you're satisfied with the error.

Newton's method - general formula

Consider $y = f(x)$ and x_0. Formula for the tangent line is given:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

This line crosses the x-axis whenever $0 = f'(x_0)(x - x_0) + f(x_0)$. In other words,

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Procedure:

- Guess x_0 which is relatively close to the solution,
- consider a recursively defined sequence:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- expect that for a large n the error $|x - x_n|$ (x is the actual solution) to be small.

Newton's method -homework

1. approximate $\sqrt[5]{7}$,
2. the function $f(x) = x^3 - 3x^2 - 3x + 6$ has a root between 3 and 4 (check signs of values at endpoints). Find a good approximation of the root using Newtons formula and divide and conquer method. Compare the algorithms.

Extreme values: introduction

Definition

Let a be a point of the domain of a function $y = f(x)$. The value $f(a)$ is called

- *local maximum* if $f(x) \leq f(a)$ for all domain points in an open interval containing a,
- *local minimum* if $f(x) \geq f(a)$ for all domain points in an open interval containing a,
- *absolute maximum* if $f(x) \leq f(a)$ for all domain points,
- *absolute minimum* if $f(x) \geq f(a)$ for all domain points.

Extreme values

Neccessary condition for local extreme values

If a function $y = f(x)$ has a local maximum or minimum at an interior point a of its domain and $f'(a)$ exists then

$$f'(a) = 0.$$

Proof..

Definition

An interior point a of the domain of a function $y = f(x)$ is called a *critical point* if $f'(c) = 0$ or $f'(c)$ doesn't exist.

Rolle's theorem and Mean Value Theorem: motivations

Question

Suppose two different functions have the same derivative; what can you say about the relationship between the two functions?

Question

Suppose you drive a car from toll booth on a toll road to another toll booth at an average speed of 150 km per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 140 km per hour speed limit?

Rolle's Theorem

Theorem

Let $y = f(x)$ be continuous at every point of a closed interval $[a, b]$ and let it be differentiable at every point of an open interval (a, b). If only $f(a) = f(b)$ then there is at least one $x_0 \in (a, b)$ for which

$$f'(x_0) = 0.$$

Proof: It follows by the fact that $f(x)$ has an extreme value for argument in (a, b).

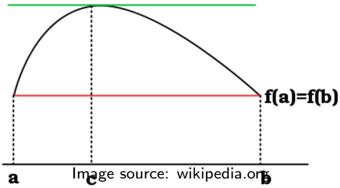


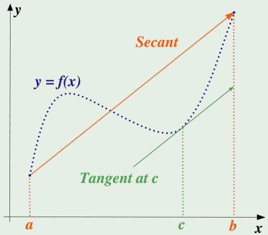
Image source: wikipedia.org

Mean Value Theorem

Mean Value Theorem

Let $y = f(x)$ be a function which is continuous at every point of $[a, b]$ and differentiable at every point of (a, b). Then there is at least one $x_0 \in (a, b)$ for which

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$



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Mean Value Theorem

Mean Value Theorem

Let $y = f(x)$ be a function which is continuous at every point of $[a, b]$ and differentiable at every point of (a, b). Then there is at least one $x_0 \in (a, b)$ for which

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Proof: Let $m = \frac{f(b)-f(a)}{b-a}$ and define $g(x) = f(x) - m \cdot (x - a) - f(a)$. Function $y = g(x)$ is differentiable and $g'(x) = f'(x) - m$. Moreover, $g(a) = g(b) = 0$. By Rolle's theorem there is $x_0 \in (a, b)$ such that $g'(x_0) = 0$. In other words, $f'(x_0) = m = \frac{f(b)-f(a)}{b-a}$.

Mean Value Theorem: Corollaries

Corollary

If $y = f(x)$ is a function whose derivative is constantly equal to 0 then f is a constant function.

Proof...

Corollary

Suppose $y = f(x)$ is continuous at every point of $[a, b]$ and differentiable at every point of (a, b). If $f'(x) > 0$ for any $x \in (a, b)$ then the function $y = f(x)$ is increasing.

Proof...

Extreme values

Sufficient condition for local extreme values

Let $y = f(x)$ be a continuous function and let a be a critical point of f. Then

- if $f'(x) > 0$ for $x < a$ and $f'(x) < 0$ for $x > a$ then f has a local maximum at a,
- if $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > a$ then f has a local minimum at a,
- if $f'(x)$ doesn't change sign around a then there is no extreme value at a.

Proof follows by 2nd corollary of MVT. Example: Consider a function $f(x) = |x| \cdot (x^2 - 3)$.

Indeterminate expressions

L'Hospital rule

Let D be a set defined by $(a, b) \setminus \{c\}$ and let $f, g : D \to \mathbb{R}$ be two functions defined on D. If $y = f(x)$ and $y = g(x)$ are differentiable at every point of D and $g'(x) \neq 0$ for any $x \in D$ and, moreover,

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \lim_{x \to c} f(x) = \pm \lim_{x \to c} g(x) = \pm\infty$$

then

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to c} \frac{f(x)}{g(x)} = L$$

Indeterminate expressions

L'Hospital rule (a different version)

Let D be a set defined by (a, ∞) and let $f, g : D \to \mathbb{R}$ be two functions defined on D. If $y = f(x)$ and $y = g(x)$ are differentiable at every point of D and $g'(x) \neq 0$ for any $x \in D$ and, moreover,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0 \text{ or } \lim_{x\to\infty} f(x) = \pm \lim_{x\to\infty} g(x) = \pm\infty$$

then

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \implies \lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

Indeterminate expressions

Remark

L'Hostpital rule is used when dealing with indeterminate expressions of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Remark

L'Hostpital rule can be applied indirectly to other indeterminate expressions:

- $0 \cdot \infty$: $f \cdot g = \frac{f}{\frac{1}{g}}$,
- $\infty - \infty$: $f - g = \frac{\frac{1}{g} - \frac{1}{f}}{\frac{1}{fg}}$,
- $0^0, \infty^0, 1^\infty$: $f^g = e^{g \ln f}$

Indeterminate expressions

Example:

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{e^x}{1}=e^0=1.$$

$$\lim_{x\to 1}x^{\frac{1}{x-1}}=\lim_{x\to 1}e^{\frac{1}{x-1}\ln x},$$

$$\lim_{x\to 1}\frac{1}{x-1}\ln x=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=1.$$

L'Hostpital rule: limitations

Warning

It may happen that

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

doesn't exist. If this is the case then we CANNOT conclude anything about

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

Consider a limit $\lim_{x \to \infty} \frac{3x - \sin x}{2x + \sin x}$.

L'Hospital rule - the controversy (ies)

Fact

Bernoulli and L'Hospital signed a contract which gave l'Hospital the right to use Bernoulli's discoveries as he pleased.

Fact

L'Hospital rule is mostly about circular reasoning.

To see this consider

$$\lim_{x\to 0} \frac{\sin x}{x} \qquad \lim_{x\to 0} \frac{e^x - 1}{x}.$$

Convexity and concavity

Definition

A function $y = f(x)$ is said to be *convex* (concave up) on an interval I if $f'(x)$ is increasing on I. A function $y = f(x)$ is said to be *concave* (concave down) on an interval I if $f'(x)$ is decreasing on I.

Example: consider $y = x^3$.

Convexity and concavity

Second derivative test for convexity

A function $y = f(x)$ is convex on an interval where $f''(x) > 0$ and is concave whenever $f''(x) < 0$.

Definition

A point a of the domain of $y = f(x)$ for which $f'(a)$ exists and around which convexity changes is called a *point of inflection*.

Theorem

If a is a point of inflection of $y = f(x)$, where f is twice differentiable, then

$$f''(a) = 0.$$

Second derivative test for extreme values

Theorem

Let a function $y = f(x)$ be twice differentiable in a neighbourhood of a point a of its domain. Then:

- if $f'(a) = 0$ and $f''(a) < 0$ then f has a local maximum at a,
- if $f'(a) = 0$ and $f''(a) > 0$ then f has a local minimum at a.